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# On a variance associated with the distribution of general sequences in arithmetic progressions. I

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# On a variance associated with the distribution of general sequences in arithmetic progressions. I

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An asymptotic formula of Montgomery–Hooley type is established for general sequences which, for relatively small moduli, are approximately equidistributed in the reduced residue classes.

**Keywords: variance; distribution; sequences; reduced residue classes**

## 1. Introduction

In this memoir we are concerned with the development of those ideas contained in Goldston & Vaughan (1996) that relate to more general situations. Thus we are following the precedent set by Hooley (1975*c*) in which his method, first applied to the primes in Hooley (1975*a, b*), is then turned and directed towards arithmetical sequences of a rather general form.

In particular, we are interested in the extent to which it is possible to obtain an asymptotic formula for the variance

$$V(x,Q)=\sum_{q\leqslant Q}\sum_{a\in\mathcal{A}(q)}|A(x;q,a)-f(q,a)\Phi(x)|^2, \tag{1.1}$$

where $\mathcal{A}(q)$ is a suitable set of residue classes modulo $q$, $A(x;q,a)$ denotes

$$A(x;q,a)=\sum_{\substack{n\leqslant x\\ n\equiv a\,(\mathrm{mod}\,q)}}a_n, \tag{1.2}$$

and $f$ and $\Phi$ appropriately reflect the local and global properties, respectively, of the real sequence $\{a_n\}$.

The simplest generalization from the primes, and one which can be very useful in practice as sequences arising from a sieving procedure often meet the requirements described below, is that in which the distribution is restricted to, and uniform within, the reduced residue classes modulo $q$, at least for relatively small $q$. Thus, for the purposes of this paper we take $\mathcal{A}(q)$ to be the set of reduced residue classes modulo $q$.

It is natural to suppose that $\Phi$ be fairly smooth, but as it stands all methods for dealing with $V(x,Q)$ require some understanding of objects such as

$$\int_0^{x-h}\Phi'(y)\Phi'(y+h)\,\mathrm{d}y.$$

Even in the case of primes this leads to some technical complications and to obviate this it is normal to consider the primes in weighted form. In order to apply weights



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properly here we make the following assumptions with regard to $\Phi$. We suppose that there is a real number $x_0 \geqslant 1$ such that on $[x_0, \infty)$ the function $\Phi$ is non-negative, has continuous second deriviative, $\Phi'(x) > 0$, $\Phi(x)\Phi''(x) \ll \Phi'(x)^2$ and $\Phi''(x) \neq 0$. We observe in passing that this implies that $\Phi(x) \ll x\Phi'(x)$ and that there is a positive number $\delta$ such that for all sufficiently large $x$ we have $\Phi(x) > x^\delta$. Thus, rather than consider $V$ in the form given above, we replace $A(x; q, a)$ by

$$B(x; q, a) = \sum_{\substack{x_0 < n \leqslant x \\ n \equiv a \,(\mathrm{mod}\, q)}} b_n, \tag{1.3}$$

where

$$b_n = \frac{a_n}{\Phi'(n)}, \tag{1.4}$$

and suppose that

$$V(x, Q) = \sum_{q \leqslant Q} \sum_{a=1}^{q} \left| B(x; q, a) - E(q, a) \frac{x}{\phi(q)} \right|^2, \tag{1.5}$$

where $E(q, a)$ is 1 or 0 according to whether $(q, a) = 1$ or $(q, a) > 1$. There would be little point in studying this form of the variance if $E(q, a)x/\phi(q)$ were not a fairly good approximation to $B(x; q, a)$, at least for smaller values of $q$. It is natural, then, to suppose that there is an increasing function $\Psi(x)$, with $\Psi(x) > \log x$ for all large $x$, $\Psi(1) > 0$ and

$$\int_1^x \Psi(y)^{-1} \, \mathrm{d}y \ll x\Psi(x)^{-1},$$

such that

$$A(x; q, a) = E(q, a) \frac{\Phi(x)}{\phi(q)} + O(\Phi(x)/\Psi(x)) \tag{1.6}$$

uniformly for all real $x \geqslant 1$ and natural numbers $q$ and $a$. Here we note that immediately from the above assumptions we have $\Psi(x) \ll x$.

With the above definitions it is now possible to state a simple conclusion.

**Theorem 1.1.** *Let*

$$U(x, Q) = V(x, Q) - Q \sum_{x_0 < n \leqslant x} b_n^2 + Qx \log \frac{x}{Q} + cxQ, \tag{1.7}$$

*where*

$$c = \gamma + \log(2\pi) + \sum_p \frac{\log p}{p(p-1)}, \tag{1.8}$$

*and suppose that for all sufficiently large* $x$ *we have* $\log x \leqslant \Psi(x) \leqslant x^{1/2}$. *Then uniformly for* $x^{2/3} \leqslant Q \leqslant x$ *we have*

$$U(x, Q) \ll Q^{3/2} x^{1/2} + \left( x^2 + x \sum_{x_0 < n \leqslant x} b_n^2 \right) (\log 2x)^{3/2} / \sqrt{\Psi(x)}. \tag{1.9}$$

By working harder, the $\sqrt{\Psi(x)}$ undoubtedly can be replaced by $\Psi(x)^\theta$ for some $\theta \in (\frac{1}{2}, 1]$, albeit an inflated logarithmic factor would be required in (1.9). It might even be possible to take $\theta = 1$.

We remark in passing that the techniques described herein would suffice just as well to treat $B(y + x; q, a) - B(y; q, a)$ provided that $y$ does not grow at an appreciably faster rate than $x$, and that then simple partial summation techniques could be applied to obtain a concomitant theorem for $A(x; q, a)$.

If one supposes further that the sequence $\{a_n\}$ is the characteristic function of a set, then it follows by partial summation from (1.6) with $q = a = 1$ that the sum over $n$ in (1.7) and (1.9) may be replaced by

$$\int_{x_0}^{x} \frac{1}{\Phi'(y)}\,\mathrm{d}y. \tag{1.10}$$

Returning to (1.7), it is interesting that when $\Psi(x)$ is at least as large as $(\log 2x)^{3+\delta}$, where $\delta$ is a positive constant, then the positivity of $V$ ensures that

$$\sum_{x_0 < n \leqslant x} b_n^2 \gg x \log \Psi(x).$$

Thus, condition (1.6) ensures that the $a_n$ cannot be the characteristic function of a dense subset of the integers.

## 2. Simple consequences of (1.6)

We need to extract several pieces of information from the basic assumption (1.6).

**Lemma 2.1.** *Let $E(q, a)$ be* 1 *or* 0 *according to whether $(q, a) = 1$ or $(q, a) > 1$. Then*

$$B(x; q, a) = E(q, a)\frac{x}{\phi(q)} + O(x/\Psi(x))$$

*uniformly for all real $x \geqslant 1$ and natural numbers $q$ and $a$.*

*Proof*. This is a straightforward application of partial summation technique. We have

$$B(x; q, a) = \left[\frac{A(y; q, a)}{\Phi'(y)}\right]_{x_0}^{x} - \int_{x_0}^{x} \left(\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{\Phi'(y)}\right)\right) A(y; q, a)\,\mathrm{d}y.$$

We apply (1.6) and observe that

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{\Phi'(y)}\right)$$

is of fixed sign. The main terms are easily handled, and the contributions from the error term in (1.6) to the above is

$$\ll \frac{\Phi(x)}{\Phi'(x)\Psi(x)} + 1 + \int_{x_0}^{x} \frac{\mathrm{d}y}{\Psi(y)}.$$

The lemma then follows easily. ■

**Lemma 2.2.** *Suppose that* $x^{1/2} < Q \leqslant x$. *Then*

$$\sum_{q\leqslant Q}\frac{1}{\phi(q)}\left(\sum_{\substack{a=1\\(a,q)=1}}^{q} B(x;q,a)-x\right) \ll \left(x+\sum_{n\leqslant x} b_n^2\right)(\log 2x)^2/\Psi(x).$$

*Proof*. We work primarily with

$$\sum_{q\leqslant Q}\frac{1}{\phi(q)}\sum_{\substack{a=1\\(a,q)=1}}^{q} B(x;q,a).$$

We invoke the definition of $B(x;q,a)$ and observe that when summed over a reduced set of residues $a$ modulo $q$ this becomes

$$\sum_{\substack{x_0<n\leqslant x\\(a,q)=1}} b_n.$$

We then interchange the order of summation of $q$ and $n$. Thus we are reduced to treating the sum

$$\sum_{\substack{q\leqslant Q\\(q,n)=1}}\frac{1}{\phi(q)}.$$

This can be done elementarily as follows. We have $1/\phi(q)=\sum_{r|q}\mu(r)^2/\phi(r)$ and so the above sum becomes

$$\sum_{\substack{r\leqslant Q\\(r,n)=1}}\frac{\mu(r)^2}{r\phi(r)}\sum_{\substack{s\leqslant Q/r\\(s,n)=1}}\frac{1}{s}.$$

The inner sum here is

$$\sum_{d|n}\frac{\mu(d)}{d}\sum_{m\leqslant Q/rd}\frac{1}{m}.$$

Here we replace the inner sum by

$$\log\frac{Q}{rd}+\gamma+O\left(\frac{rd}{Q}\right).$$

This leads to the estimate

$$\sum_{\substack{q\leqslant Q\\(q,n)=1}}\frac{1}{\phi(q)}=\sum_{\substack{r=1\\(r,n)=1}}^{\infty}\sum_{d|n}\frac{\mu(d)}{d}\left(\log\frac{Q}{rd}+\gamma\right)+O\left(\frac{\log Q}{Q}(\log 2n+d(n))\right).$$

Simple manipulations then show that the main terms here are

$$C_1\left(\log Q+C_2+\sum_{p|n}f(p)\right)\sum_{r|n}g(r),$$

where

$$C_1 = \sum_{r=1}^{\infty} \frac{\mu(r)^2}{r\phi(r)},$$

$$C_2 = \gamma + \sum_p \frac{\log p}{p^2 - p + 1},$$

$$f(p) = \frac{\log p}{p-1} - \frac{\log p}{p^2 - p + 1},$$

and $g(r)$ is multiplicative, equals 0 unless $r$ is square-free, and satisfies

$$g(p) = -\frac{p}{p^2 - p + 1}.$$

In passing we observe that in the special case $n = 1$ we have

$$\sum_{q \leqslant Q} \frac{1}{\phi(q)} = C_1(\log Q + C_2) + O\left(\frac{\log Q}{Q}\right). \tag{2.1}$$

We apply the general case and obtain

$$\sum_{q \leqslant Q} \frac{1}{\phi(q)} \sum_{\substack{a=1\\(a,q)=1}}^{q} B(x; q, a) = U + V,$$

where

$$U = \sum_{x_0 < n \leqslant x} b_n C_1 \left(\log Q + C_2 + \sum_{p|n} f(p)\right) \sum_{r|n} g(r)$$

and

$$V \ll \sum_{n \leqslant x} |b_n| \frac{\log Q}{Q} (\log 2n + d(n)).$$

By the case $q = 1$ of lemma 2.1 we have

$$\sum_{n \leqslant x} b_n \gg x,$$

when $x$ is sufficiently large. Hence, by Cauchy's ineqality we have both

$$\sum_{n \leqslant x} b_n^2 \gg x$$

and

$$\sum_{n \leqslant x} |b_n| \ll \sum_{n \leqslant x} b_n^2.$$

Therefore

$$V \ll \sum_{n \leqslant x} b_n^2 / \Psi(x).$$

With regard to $U$ we have

$$U = C_1(\log Q + C_2) \sum_{r \leqslant x} g(r) \sum_{\substack{x_0 < n \leqslant x\\ r|n}} b_n + C_1 \sum_{p \leqslant x} f(p) \sum_{r \leqslant x} g(r) \sum_{\substack{x_0 < n \leqslant x\\ [p,r]|n}} b_n.$$

Again by lemma 2.1, the contribution from each sum over $n$ is $\ll x/\Psi(x)$, except when $r = 1$ in the first sum, in which case there is an extra contribution of $x$. Therefore

$$U = xC_1(\log Q + C_2) + O(x(\log x)^2/\Psi(x)).$$

By (2.1) the main term here can be replaced by

$$x \sum_{q\leqslant Q} \frac{1}{\phi(q)}$$

and then the lemma follows at once. ■

It is useful to define here

$$G(\alpha) = \sum_{x_0<n\leqslant x} b_n e(\alpha n) \tag{2.2}$$

and

$$J(\beta) = \sum_{n\leqslant x} e(\alpha n) \tag{2.3}$$

where, as usual, $e(\cdot) = \exp(2\pi \mathrm{i}\cdot)$. Then the next lemma follows easily be partial summation.

**Lemma 2.3.** *Suppose that* $(q, a) = 1$ *and* $\alpha = a/q + \beta$. *Then*

$$G(\alpha) = \frac{\mu(q)}{\phi(q)} J(\beta) + O((1 + x|\beta|)qx/\Psi(x)).$$

## 3. Preliminary arrangements

We have

$$V(x, Q) = 2S_1 - S_2 + S_3 + 2E + O\left(\sum_{n\leqslant x} b_n^2\right)$$

where

$$S_1 = \sum_{q\leqslant Q} \sum_{n\leqslant x} \sum_{\substack{m<n \\ q\mid n-m}} b_m b_n, \tag{3.1}$$

$$S_2 = \sum_{q\leqslant Q} \frac{x^2}{\phi(q)}, \tag{3.2}$$

$$S_3 = Q \sum_{m\leqslant x} b_m^2, \tag{3.3}$$

$$E = \sum_{q\leqslant Q} \frac{x}{\phi(q)} \left(x - \sum_{\substack{a=1 \\ (a,q)=1}}^{q} B(x; q, a)\right). \tag{3.4}$$

Lemma 2.2 enables us to bound $E$. Thus

$$V(x, Q) = 2S_1 - S_2 + S_3 + O\left(\left(x^2 + x\sum_{n\leqslant x} b_n^2\right)\frac{(\log 2x)^2}{\Psi(x)}\right). \tag{3.5}$$

As is usual in these questions, the main part of our argument is concerned with the sum $S_1$.

For future reference observe that it may be supposed that $x$ is sufficiently large.

## 4. The Farey dissection and an exponential sum

This section largely imitates § 3 of Goldston & Vaughan (1996). Let

$$F(\alpha) = \sum_{q \leqslant Q} \sum_{r \leqslant x/q} e(\alpha q r), \tag{4.1}$$

and $G$ be as in (2.2). Then, by (3.1),

$$S_1 = \int_0^1 F(\alpha)|G(\alpha)|^2 \, \mathrm{d}\alpha. \tag{4.2}$$

For any arbitrary function $F : \mathbb{N} \to \mathbb{C}$ we have

$$\sum_{q \leqslant Q} \sum_{r \leqslant x/q} f(qr) = \sum_{q \leqslant \sqrt{x}} \sum_{r \leqslant x/q} f(qr) + \sum_{r \leqslant \sqrt{x}} \sum_{\sqrt{x} < q \leqslant \min(Q, x/r)} f(qr).$$

For convenience an expression of the kind on the right-hand side is written in the form

$$\sum_{l \leqslant \sqrt{x}} \Bigg( \sum_{m \leqslant x/l} + \sum_{\sqrt{x} < m \leqslant \min(Q, x/l)} \Bigg) f(lm). \tag{4.3}$$

Thus

$$F(\alpha) = F_q(\alpha) + H_q(\alpha), \tag{4.4}$$

where

$$F_q(\alpha) = \sum_{\substack{l \leqslant \sqrt{x} \\ q \mid l}} \Bigg( \sum_{m \leqslant x/l} + \sum_{\sqrt{x} < m \leqslant \min(Q, x/l)} \Bigg) e(\alpha l m) \tag{4.5}$$

and $H_q(\alpha)$ is the corresponding multiple sum with $q \nmid l$. On performing the inner summation in $H_q(\alpha)$ we have

$$H_q(\alpha) \ll \sum_{\substack{l \leqslant \sqrt{x} \\ q \nmid l}} \min(xl^{-1}, ||\alpha l||^{-1}).$$

For a given $a$ and $q$ with $(a, q) = 1$ we put

$$\beta = \alpha - a/q.$$

Then

$$||\alpha l|| \geqslant ||al/q|| - |\beta| l$$

and so when

$$l \leqslant \sqrt{x}, \quad q \nmid l \quad \text{and} \quad |\beta| \leqslant \tfrac{1}{2} q^{-1} x^{-1/2}$$

we have

$$H_q(\alpha) \ll \sum_{\substack{l \leqslant \sqrt{x} \\ q \nmid l}} ||al/q||^{-1} \ll (\sqrt{x} q^{-1} + 1) q \log 2q$$

and so

$$H_q(\alpha) \ll (\sqrt{x} + q) \log 2q \quad \text{when} \quad |\beta| \leqslant \tfrac{1}{2} q^{-1} x^{-1/2}. \tag{4.6}$$

We suppose that $R$ satisfies

$$2\sqrt{x} \leqslant R \leqslant \tfrac{1}{2}x \tag{4.7}$$

and consider a typical interval $\mathfrak{M}(q,a)$ associated with the element $a/q$ of the Farey dissection of order $R$, namely, when $1 \leqslant a \leqslant q \leqslant R$ and $(a,q)=1$,

$$\mathfrak{M}(q,a) = \left(\frac{a+a_-}{q+q_-}, \frac{a+a_+}{q+q_+}\right],$$

where $q_\pm$ is defined by $aq_\pm \equiv \mp 1 \pmod q$ and $R - q < q_\pm \leqslant R$ and $a_\pm$ is defined by $a_\pm = (aq_\pm \pm 1)/q$. We observe that

$$\left|\frac{a+a_\pm}{q+q_\pm} - \frac{a}{q}\right| = \frac{1}{q(q+q_\pm)}$$

lies in $[1/(2qR), 1/(qR))$.

We have

$$S_1 = \sum_{q\leqslant R} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathfrak{M}(q,a)} F(\alpha)|G(\alpha)|^2 \,\mathrm{d}\alpha$$

and

$$\sum_{q\leqslant R} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathfrak{M}(q,a)} H_q(\alpha)|G(\alpha)|^2 \,\mathrm{d}\alpha \ll R(\log x)\int_0^1 |G(\alpha)|^2 \,\mathrm{d}\alpha.$$

We note also that $F_q(\alpha) = 0$ when $q > \sqrt{x}$. Hence

$$S_1 = S_4 + O\left(R\log x \sum_{n\leqslant x} b_n^2\right), \tag{4.8}$$

where

$$S_4 = \sum_{q\leqslant\sqrt{x}} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathfrak{M}(q,a)} F_q(\alpha)|G(\alpha)|^2 \,\mathrm{d}\alpha. \tag{4.9}$$

Let $\beta = \alpha - a/q$. Then, by (4.5),

$$F_q(\alpha) \ll \sum_{m\leqslant\sqrt{x}/q} \min(xq^{-1}m^{-1}, ||\beta qm||^{-1}). \tag{4.10}$$

Hence

$$F_q(\alpha) \ll \frac{x\log(2\sqrt{x}/q)}{q+qx|\beta|} \quad (q\leqslant\sqrt{x}, |\beta| \leqslant \tfrac{1}{2}q^{-1}x^{-1/2}). \tag{4.11}$$

Suppose that $q \leqslant \sqrt{x}$ and define the major arc $\mathfrak{N}(q,a)$ by

$$\mathfrak{N}(q,a) = \left[\frac{a}{q} - q^{-1}(2R)^{-1}, \frac{a}{q} + q^{-1}(2R)^{-1}\right]. \tag{4.12}$$

Then $\mathfrak{N}(q,a) \subset \mathfrak{M}(q,a)$ and for $\alpha \in \mathfrak{M}(q,a)\backslash\mathfrak{N}(q,a)$ we have

$$F_q(\alpha) \ll R\log x.$$

Moreover, the same conclusion holds when $\alpha \in \mathfrak{N}(q,a)$ and $q > x/R$. Thus, by (4.8) and (4.9),

$$S_1 = S_5 + O\biggl(R(\log x)\sum_{n\leqslant x} b_n^2\biggr), \tag{4.13}$$

where

$$S_5 = \sum_{q\leqslant x/R}\ \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathfrak{N}(q,a)} F_q(\alpha)|G(\alpha)|^2\,\mathrm{d}\alpha. \tag{4.14}$$

## 5. The major arcs

We do not dwell for long on the major arcs $\mathfrak{N}(q,a)$ as the method is almost completely standard. In view of (4.7), on each arc, if necessary, we may invoke (4.11). Moreover, by lemma 2.3,

$$F_q(\alpha)|G(\alpha)|^2 = \frac{\mu(q)^2}{\phi(q)^2}F_q(\beta)|J(\beta)|^2 + \Delta_1 + \Delta_2,$$

where

$$\Delta_1 \ll \frac{x^3\log x}{(1+x|\beta|)\phi(q)\Psi(x)},$$
$$\Delta_2 \ll x^3q(\log x)(1+x|\beta|)\Psi(x)^{-2},$$

and, as usual, $\alpha = a/q+\beta$. Hence, by (4.12) and (4.14),

$$S_5 = S_6 + O(x^3(\log x)^2R^{-1}\Psi(x)^{-1} + x^5(\log x)R^{-3}\Psi(x)^{-2}), \tag{5.1}$$

where

$$S_6 = \sum_{q\leqslant x/R}\frac{\mu(q)^2}{\phi(q)}\int_{I(q)} F_q(\beta)|J(\beta)|^2\,\mathrm{d}\beta, \tag{5.2}$$

with

$$I(q) = [-\tfrac{1}{2}q^{-1}R^{-1}, \tfrac{1}{2}q^{-1}R^{-1}]. \tag{5.3}$$

We now wish to replace each $I(q)$ by a unit interval. It is desirable to make as much use of (4.11) as possible, but then conditions there necessitate proceeding in two stages. By (4.11), when

$$1/(2qR) \leqslant |\beta| \leqslant 1/(2q\sqrt{x}),$$

the contribution from the integrand is

$$\ll \frac{\log x}{q\phi(q)^2|\beta|^3},$$

and, by the crude estimate

$$F_q(\beta) \ll xq^{-1}\log x$$

stemming from (4.10), when $1/(2q\sqrt{x}) \leqslant |\beta| \leqslant \frac{1}{2}$ it is

$$\ll \frac{x^3\log x}{q\phi(q)^2(1+x|\beta|)^2}.$$

We may also use this latter bound to estimate the contribution when we add in the $q$ in the range $x/R < q \leqslant \sqrt{x}$. Thus, by (5.2),

$$S_6 = S_7 + O(xR\log x + x^{3/2}(\log x)^2)$$

where

$$S_7 = \sum_{q\leqslant\sqrt{x}} \frac{\mu(q)^2}{\phi(q)} \int_{-1/2}^{1/2} F_q(\beta)|J(\beta)|^2\,\mathrm{d}\beta.$$

Thus, by (4.13) and (5.1), $S_1$ differs from $S_7$ by an amount which is

$$\ll R\log x \sum_{n\leqslant x} b_n^2 + xR\log x + x^{3/2}(\log x)^2 + x^3(\log x)^2 R^{-1}\Psi(x)^{-1} + x^5(\log x)R^{-3}\Psi(x)^{-2}.$$

The optimal choice for $R$ here is

$$R = x(\log x)^{1/2}\Psi(x)^{-1/2}.$$

Hence

$$S_1 = S_7 + O\biggl(\biggl(x^2 + x\sum_{n\leqslant x} b_n^2\biggr)(\log x)^{3/2}\Psi(x)^{-1/2}\biggr). \qquad (5.4)$$

## 6. Completion of the proof of theorem 1.1

The completion of the proof now closely shadows that of Goldston & Vaughan (1996), there being minor detail changes because we do not desire to invoke the Riemann hypothesis. Following the details of § 7 of that paper (Goldston & Vaughan 1996) we find first that, via (5.4),

$$S_1 = \tfrac{1}{2}x^2 \sum_{l\leqslant\sqrt{x}} \frac{1}{\phi(l)} + \tfrac{1}{2}xW(\sqrt{x}) - \tfrac{1}{2}Q^2W(x/Q) + O\biggl(\biggl(x^2 + x\sum_{n\leqslant x} b_n^2\biggr)(\log x)^{3/2}\Psi(x)^{-1/2}\biggr). \qquad (6.1)$$

Then, by (3.1)–(3.3) and (3.5),

$$V(x,Q) = Q\sum_{n\leqslant x} b_n^2 + xW(\sqrt{x}) - Q^2W(x/Q) - C_1x^2\log\frac{Q}{\sqrt{x}} + O\biggl(\biggl(x^2 + x\sum_{n\leqslant x} b_n^2\biggr)(\log x)^{3/2}\Psi(x)^{-1/2}\biggr), \qquad (6.2)$$

where

$$C_1 = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \qquad (6.3)$$

and

$$W(X) = \sum_{l\leqslant X} \frac{1}{\phi(l)}(X-l)^2. \qquad (6.4)$$

Let

$$C_2 = \left(\gamma - \frac{3}{2} - \sum_p \frac{\log p}{p^2 - p + 1}\right) C_1, \tag{6.5}$$

$$C_3 = \gamma + \log 2\pi + \sum_p \frac{\log p}{p(p-1)}, \tag{6.6}$$

and define

$$E(X) = W(X) - C_1 X^2 \log X - C_2 X^2 - X \log X - C_3 X. \tag{6.7}$$

Then we would like to invoke lemma 5 of Goldston & Vaughan (1996) to bound $E(X)$, but the estimate given there depends on the Riemann hypothesis. However, quite plainly, without any hypothesis it is still possible to move the relevant contour to the line $\Re s = -\frac{3}{2}$ using only standard estimates for the Riemann zeta function (see, for example, §§ 3.5 and 3.11 of Titchmarsh (1986)). Thus we obtain

$$E(X) = O(X^{1/2}). \tag{6.8}$$

Indeed a small further saving could be made by penetrating the zero free region in the critical strip of the Riemann zeta function, but this is not of great importance.

Combining (6.2), (6.7) and (6.8) now immediately gives theorem 1.1.

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